Statistics of trees and branched polymers from a generalised Hilhorst model

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Abstract. A generalisation of the Hilhorst model in which at each site, x, on a lattice, there is an n-state variable $\nu(x)$, and an s-state variable, $\sigma(x)$, which interact via a Hamiltonian

$$\mathcal{H} = -nK \sum_{\langle x, x' \rangle} \delta^{\nu(x)\nu(x')}(s\delta^{\sigma(x)\sigma(x')} - 1) - h \sum_x \delta^{\nu(x)1} (s\delta^{\sigma(x)1} - 1)$$

is introduced. It is shown that if $(s-1) = \lambda n$, the $n = 0$ limit of the partition function for this model is the generating function for trees in which $\ln K$ is the chemical potential for bonds (monomers), $\ln \lambda$ for the number of trees (polymers) and $\ln h$ for the number of free ends of all trees. Fields which mark any point and fields which mark only external points of a polymer are identified. The above Hamiltonian is converted to a field theory which is used to discuss the dependence on the monomer number, N, of critical properties such as the radius of gyration of branched polymers with a small number of branchings. It is shown that these properties are controlled by the usual $n = 0$ polymer fixed point.

1. Introduction

The solutions to a number of non-thermodynamic statistical problems may be obtained from the thermodynamics and correlation functions of certain model Hamiltonians which are functions of some parameter that is analytically continued to an apparently non-physical value. The most familiar of these problems is that of the self-avoiding random walk (SAW) which is equivalent to the centrifugal statistics of a single isolated long polymer. de Gennes (1972) showed that the statistics of SAW's could be generated by considering the $n = 0$ limit of the model spin Hamiltonian $\mathcal{H} = -J\sum_{\langle x, x' \rangle} s(x) \cdot s(x')$ where $s(x)$ is an n-component classical vector of fixed length and x and x' are nearest-neighbour sites on a lattice. This result has been generalised by des Cloizeaux (1975) to permit a treatment of semi-dilute solutions of long polymers. Predictions based on this theory are in good agreement with experiments (Daoud *et al* 1975). A fairly substantial theoretical literature of polymer statistics using the $n = 0$ n-vector model (Emery 1975, Jasnow and Fisher 1976, Daoud and Jannink 1976, de Gennes 1975, Burch and Moore 1976a, b, Moore 1977, Schaffer and Witten 1977) has been developed.

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Another model Hamiltonian which is relevant to a variety of non-thermodynamic statistical problems is the s-state Potts model (Potts 1952). In this model there is a variable $\sigma(x)$ at each site x which can exist in any of s different states. Bonds connecting neighbouring sites have one energy if the sites are in the same state and another energy if they are in different states. The Hamiltonian is then

$$\mathcal{H} = -sJ \sum_{\langle x,x' \rangle} \delta^{\sigma(x)\sigma(x')}. \tag{1.1}$$

The one-state limit of this model describes the statistics of percolating clusters (Kasteleyn and Fortuin 1969, Fortuin and Kasteleyn 1972, Harris *et al* 1975). The zero-state limit, on the other hand, counts trees (Stephen 1976) on a lattice (i.e. configurations with no closed loops). Furthermore, the coefficient of $1/J$ in the susceptibility $\chi(x, x')$ in the large-J limit is proportional to the resistance between points x and x' of a network in which each bond is occupied by a resistance (Fortuin and Kasteleyn 1972, Dasgupta *et al* 1978).

Recently, Hilhorst (1976, 1977) introduced a model with discrete symmetry (as opposed to the continuous symmetry of the n-vector model) that can be used to generate the statistics of self-avoiding walks. This model can be employed in low-dimensional position-space renormalisation-group calculations (For a review of position-space renormalisation-group transformations see van Leeuwen 1975) whereas the n-vector model cannot. In the Hilhorst model, there is a vector $\boldsymbol{\mu}(x)$ at each site that is constrained to point along the positive and negative axes of an n-dimensional hypercube: $\boldsymbol{\mu}(x) = (0, \ldots, 0, \pm\sqrt{n}, \ldots)$. The Hamiltonian is then

$$\mathcal{H} = -J \sum_{\langle x,x' \rangle} \boldsymbol{\mu}(x) \cdot \boldsymbol{\mu}(x'). \tag{1.2}$$

saw's are generated by the $n = 0$ limit of this model. An alternate form of this Hamiltonian allows one to see how saw's are obtained if one knows that the zero-state Potts model (Potts 1952) generates tree-like configurations:

$$\mathcal{H} = -nJ \sum_{\langle x,x' \rangle} \delta^{\nu(x)\nu(x')} \sigma(x)\sigma(x') \tag{1.3}$$

where $\nu(x)$ is an n-state variable and $\sigma(x)$ takes on the values ± 1. Now, consider the correlation function

$$G(x, x') = \langle \boldsymbol{\mu}(x) \cdot \boldsymbol{\mu}(x') \rangle = \langle \delta^{\nu(x)\nu(x')} \sigma(x)\sigma(x') \rangle. \tag{1.4}$$

The $n \to 0$ limit allows only tree-like configurations to appear in G, whereas the fact that $\Sigma_{\sigma=\pm1}\sigma = 0$ eliminates all configurations with end points at sites other than x and x'. Thus the $n \to 0$ limit of $G(x, x')$ counts the number of saw's beginning at x and ending at x'.

In this paper, we introduce a generalisation of the Hilhorst model in which $\sigma(x)\sigma(x')$ is replaced by $(s\delta^{\sigma(x)\sigma(x')} - 1)$ where $\sigma(x)$ is an s-state variable rather than a two-state variable. This model, which we will call the ns-component Hilhorst model, is related to an ns-state Potts model. $G(x, x')$ for this model still generates saw's when $n \to 0$ since $\Sigma_\sigma(s\delta^{\sigma\sigma'} - 1) = 0$. The generalisation to s arbitrary rather than $s = 2$ will permit a rather comprehensive treatment of the statistics of branched polymers and trees as well as linear polymers when s is analytically continued to one. The principal purpose of this paper is to explore how one might obtain generating

functions for the statistics of branched polymers using the generalised Hilhorst Hamiltonian, and how this Hamiltonian can be converted into a field theory suitable for study using analytic renormalisation-group procedures (Wilson and Kogut 1974). In future publications, we will use the ns-component Hilhorst model to obtain actual calculations of statistical properties of branched polymers.

In order to understand the limitations of the ns-component Hilhorst model, it is useful to distinguish between polymers and trees. A tree is any connected graph with no closed loops. Examples of trees on a lattice with number of branches $N^B = 1, 3$ and 4 are shown in figure 1. We will be interested in averages over ensembles in which the
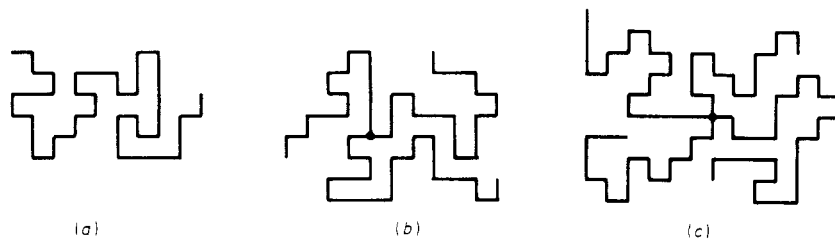
Figure 1. Examples of trees on a lattice. (a) shows a tree with a single branch. This is equivalent to a self-avoiding walk between its end points. (b) and (c) show trees with three and four branches.

average lengths of all branches are equal. In this case, we can represent trees by the more schematic graphs shown in figure 2. The external vertices (marked, for example, by a, b and c in fig. $2(a)$) of these trees are indistinguishable. A real branched polymer consists of distinguishable linear chains connected by poly-functional units allowing fairly unconstrained rotation about their axes. For example, the chains Aa and Ab in figure $2(a)$ can rotate about the axis Ac leading to a distinct configuration of the polymer if a and b are interchanged. Furthermore, in a solution (in three dimensions), a real molecule rotates and tumbles freely so that cyclic permutations of the three external vertices lead to distinct configurations of the polymer. There are, therefore, $6(=3\times 2)$ polymeric configurations corresponding to the tree graph figure $2(a)$, and $8(=2\times 4)$ configurations corresponding to that of figure $2(b)$.

A polymer of *type* T can be placed into one-to-one correspondence with a tree graph T. In general there are N^T ways that such a polymer can be accommodated into its associated tree graph. Examples of trees with tri-functional branching units are
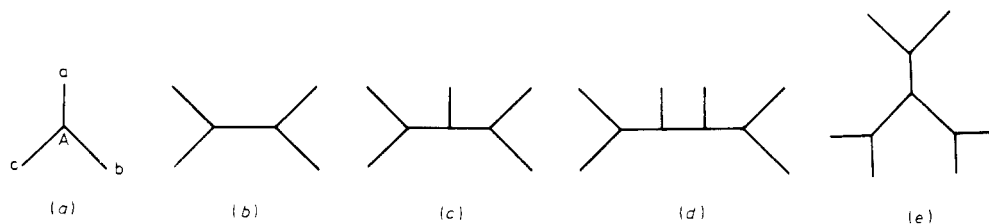
Figure 2. Schematic graphs of trees or branched polymers in which each branch has the same length on the average.

shown in figure 2. Notice that there are two different tree graphs with $N^B = 9$ (figures $2(d)$ and $2(e)$) with widely different values of N^T (8 for figure $2(d)$ and $48(= 3! \times 2^3)$ for figure $2(e)$) whereas for $N^B \le 7$, there is only one tree graph for each value of N^B.

The model presented here deals most naturally with trees. It is obvious, however, from the above considerations that the configurational statistics of a branched polymer of type T can be obtained trivially from that of the corresponding tree. In particular, we will be able to calculate such quantities as the number of configurations of a polymer with N_b monomers and three branches with end points at x_1, x_2 and x_3 and to distinguish this from the number of configurations of a linear polymer passing through x_2 with end points at x_1 and x_3. We will also be able to calculate such quantities as the radius of gyration, $\langle R_G^2 \rangle$, of a branched polymer of type T. Detailed calculations of $\langle R_G^2 \rangle$, which will be presented in a later publication, proceed more easily for $N^B \le 7$ when only bi- and tri-functional units are present.

The above considerations refer to the properties of a single type of tree or polymer averaged over an ensemble in which each branch has the same number of monomers on the average. The situation becomes more complex if one wants to inquire about properties such as the radius of gyration averaged over ensembles in which the number of branches and type of polymer are not determined. In this case, different probabilities of occurrence for each type of tree or polymer can be specified. For example, one might consider an ensemble in which each configuration, independent of type, of a tree with N^B branches occurs with equal probability. The generating function for this ensemble will turn out to be an appropriate limit of the partition function of the model presented here. Thus, we will be able to discuss properties of dilute and semi-dilute solutions of trees in this ensemble. In real polymer systems, the probability distribution usually depends on the kinetics of the formation of the polymer molecule. A particular model (Flory 1941) of such kinetics leads to a statistical ensemble that has been studied by Zimm and Stockmayer (1949) and by de Gennes (1968). This statistical ensemble can also be treated using the present model. Detailed discussions of properties of polymers and trees in these two ensembles will be presented in a future publication. In particular, we will generalise the calculations of Zimm and Stockmayer (1949) and of de Gennes (1968) of the radius of gyration of branched molecules to include a repulse interaction between monomer units.

This paper has four sections in addition to the introduction. Section 2 presents the generalised Hilhorst model and some necessary definitions. Section 3 shows how the model can yield information about the statistics of: (a) linear polymers; and (b) trees and branched polymers. Section 4 shows how the generalised Hilhorst model can be converted into a continuum field theory permitting calculations of exponents and scaling functions in $4-\epsilon$ dimensions. Finally, § 5 summarises the results of this paper.

2. Model and definitions

Consider a d-dimensional hypercubic lattice with L sites labelled by x. At each site, there is an n-state variable, $\nu(x)$, and an s-state variable, $\sigma(x)$. The generalisation of the Hilhorst Hamiltonian, H, that we consider in this paper is

$$\frac{H}{T} \equiv \mathcal{H} = -nK \sum_{\langle x,x' \rangle} \delta^{\nu(x)\nu(x')} (s\delta^{\sigma(x)\sigma(x')} - 1) - \sum_{x,\nu} \tilde{H}_\nu(x)\tau_a^\nu(x) - \sum_{x,\nu} \tilde{h}_\nu(x)\tau_b^\nu(x) - \sum_{x,\nu} f_\nu(x)$$

(2.1)

where T is the temperature, $\langle x, x'\rangle$ signifies the bond associated with nearest-neighbour sites x and x'; $\tilde{H}_\nu(x)$, $\tilde{h}_\nu(x)$ and $f_\nu(x)$ are external fields and

$$\tau_a^\nu(x) = \delta^{\nu(x)\nu}(s\delta^{\sigma(x)1} - 1) \tag{2.2a}$$

$$\tau_b^\nu(x) = \frac{1}{n-1}(n\delta^{\nu(x)\nu} - 1). \tag{2.2b}$$

It will be convenient to introduce a third field $\tau_c^\nu(x)$ via

$$\tau_c^\nu(x) = \tau_a^\nu(x) + \tau_b^\nu(x). \tag{2.3}$$

In what follows, an alternative representation of equation (2.1) will be useful. Introduce vectors $a_i^\nu = (n)^{1/2}\delta_i^\nu$ and e_l^σ satisfying

$$\sum_{i=1}^n a_i^\nu a_i^{\nu'} = n\delta^{\nu\nu'}; \qquad \sum_{\nu=1}^n a_i^\nu a_j^\nu = n\delta_{ij} \tag{2.4a}$$

$$\sum_{l=1}^{s-1} e_l^\sigma e_l^{\sigma'} = (s\delta^{\sigma\sigma'} - 1); \qquad \sum_{\sigma=1}^s e_l^\sigma = 0; \qquad \sum_{\sigma=1}^s e_l^\sigma e_{l'}^\sigma = s\delta_{ll'}. \tag{2.4b}$$

The vectors e_l^σ have been used extensively in studies of the s-state Potts model (Zia and Wallace 1975). $\mathcal{H}$ assumes a quadratic form in terms of these variables:

$$\mathcal{H} = -K\sum_{\langle x,x'\rangle} Q_{i_l}(s)Q_{i_l}(x') - \sum_x H_{i_l}(x)Q_{i_l}(x) - \sum_{x,\nu}\tilde{h}_\nu(x)\tau_b^\nu(x) - \sum_{x,\nu} f_\nu(x) \tag{2.5}$$

where the summation convention on repeated Latin subscripts is understood and where

$$Q_{i_l}(x) = a_i^{\nu(x)} e_l^{\sigma(x)}, \tag{2.6a}$$

and

$$H_{i_l}(x) = \frac{1}{n}\sum_\nu \tilde{H}_\nu(x)a_i^\nu e_l^1 \equiv \frac{1}{(n)^{1/2}}\tilde{H}_i(x)e_l^1. \tag{2.6b}$$

The goal of this paper is to evaluate the partition function,

$$Z\{K, \{\tilde{H}_\nu(x)\}, \{\tilde{h}_\nu(x)\}, \{f_\nu(x)\}) = \left(\frac{1}{ns}\right)^L \sum_{\{\nu(x),\sigma(x)\}} e^{-\mathcal{H}}, \tag{2.7}$$

as a power series in K and functions of $\tilde{H}_\nu(x)$, $\tilde{h}_\nu(s)$ and $f_\nu(x)$ and to identify each term in the series with particular tree or polymer configurations on a lattice. For some purposes, it is convenient to expand Z directly in terms of $\tilde{H}_\nu$, $\tilde{h}_\nu$ and f_ν. For other purposes, however, it is expedient to expand Z in terms of variables which can occur no more than once at any site x. In Appendix 1 we show that it is possible to choose the fields $\tilde{H}_\nu(x)$, $\tilde{h}_\nu(x)$ and $f_\nu(x)$ so that

$$e^{-\mathcal{H}} = e^{-\mathcal{H}_B}\prod_{x,\nu}[(1 - g_\nu^a(x)\tau_a^\nu(x))(1 + g_\nu^b(x)\tau_b^\nu(x))(1 - g_\nu^c(x)\tau_c^\nu(x))] \tag{2.8}$$

where $\mathcal{H}_B$ is the first term on the right-hand side of equation (2.1). We will often choose $\nu = 1$. We, therefore, write

$$g_1^\alpha(x) \equiv g_\alpha(x) \quad\text{and}\quad \tau_\alpha^1(x) = \tau_\alpha(x) \qquad \alpha = a, b, c. \tag{2.9}$$

Each term in the Taylor series expansion of Z can be represented by a graph, $\mathscr{G}$, on the lattice of L sites. Before proceeding with the evaluation of Z, it is convenient to introduce some definitions.

(1) A *bond* or *monomer* is a straight line connecting two neighbouring lattice sites. Each factor of $nK\delta^{\nu\nu'}(s\delta^{\sigma\sigma'}-1)$ in the expansion of Z is represented by a bond.

(2) A *marked* site or *vertex* is any vertex x at which a factor $\tau_a(x)$ or $\tau_c(x)$ appears. An α-*marked* vertex is a vertex at which a factor $\tau_\alpha(x)(\alpha = a \text{ or } c)$ appears. It is possible to consider the general case in which some of the marked vertices are a-marked and the others are c-marked. However, for simplicity, we will confine our attention to the case where only one type of marking (a *or* c) is present. If Z is to be expanded in powers of g_a or g_c, then a given vertex can be marked no more than once by any of these fields. On the other hand, if Z is to be expanded in powers of $\tilde{H}_1$, multiply a-marked vertices will appear. On occasion, we will refer to marked vertices as *end points*.

(3) A *cluster* is any set of vertices connected with respect to bonds. Isolated points are considered clusters. The number of clusters in a given graph will be denoted by $N_c(\mathscr{G})$.

(4) A *marked cluster* is any cluster containing marked vertices. The number of marked clusters will be denoted by $N_M(\mathscr{G})$.

(5) A *tree* is any marked cluster with no closed loops. The number of trees will be denoted by $N_t(\mathscr{G})$.

(6) An *external vertex* of a cluster is any vertex connected to the rest of the cluster by a single bond.

(7) An *internal vertex* is any other vertex in a cluster.

(8) An *a-vertex* is a vertex marked only with a-markers or not at all.

(9) A *b-vertex* is a vertex at which a factor $\tau_b(x)$ appears.

(10) The *order* of a vertex is the number of bonds entering, plus the number of markers at that vertex.

(11) A *linear polymer* is a tree containing only vertices of order two.

(12) A *branched tree* is a tree containing at least one vertex of order three or higher.

Examples of linear and branched trees are shown in figures 1 and 2. Note that the above definitions imply that a twice-marked isolated site is considered a linear polymer. The $n = 0$ n-vector model also counts this as a polymer. Note also that if Z is expanded in terms of $\tilde{H}_1$ or g_a, a linear cluster marked at two end points and any intermediate point is considered a branched tree. We will see shortly that this is not the case for c-*marked* trees when $s \to 1$.

We are now in a position to evaluate the contribution of an arbitrary graph to Z. Consider a graph containing N_b bonds, N_c clusters and N_M marked clusters. The contribution of this graph to Z can be expressed in the form,

$$n^{-L}(nK)^{N_b}n^{N_c}n^{-N_M}P(s, \mathscr{G})R(\{\tilde{H}_\nu\}, \{\hat{h}_\nu\}, \{f_\nu\}), \tag{2.10}$$

where $P(s, \mathscr{G})$ is a factor, which we will evaluate shortly, which depends only on s and the particular graph $\mathscr{G}$ and where $R(\{\tilde{H}_\nu\}, \{\hat{h}_\nu\}, \{f_\nu\})$ includes all factors depending on $\tilde{H}_\nu$, $\hat{h}_\nu$ and f_ν. Using the fundamental lattice relation (Domb 1974)

$$N_b + N_c - L = N_L \tag{2.11}$$

where N_L is the number of closed loops, we see that equation (2.10) is proportional to $n^{N_L - N_M}$. Thus in the limit $n \to 0$, the leading graphs contain no closed loops, and we

need only consider graphs consisting of trees and isolated unmarked points. Furthermore, we can replace N_M by N_t. In this case, $P(s, \mathcal{G})$ can be evaluated without too much difficulty. First, let us consider the case where only a-vertices are present. With each a-vertex of order p, there is associated a factor

$$\lambda^p_{l_1 \ldots l_p} = \sum_{\sigma=1} e^\sigma_{l_1} \ldots e^\sigma_{l_p} \tag{2.12}$$

since each bond carries a factor $s\delta^{\sigma\sigma'} - 1 = \Sigma_l e^\sigma_l e^{\sigma'}_l$. A graph must be a scalar in the indices $l_1, \ldots, l_p$. A scalar is obtained from collections of vertices carrying factors of the form of equation (2.12): (1) by marking with external fields carrying factors e^1_l; and (2) by connecting adjacent vertices by bonds. It is obvious that all external vertices must be marked. Internal vertices may or may not be marked. A pth order a-vertex marked p times contributes a factor

$$\lambda^p_{l_1 \ldots l_p} e^1_{l_1} \ldots e^1_{l_p} = (s-1)[(s-1)^{p-1} - (-1)^{p-1}]. \tag{2.13}$$

Adjacent p- and m-order a-vertices connected by a single bond lead to a new effective vertex function as follows:

$$\lambda^p_{l_1, l_2 \ldots l_p} \lambda^m_{l_1, l'_2 \ldots l'_m} = s\lambda^{p+m-2}_{l_2 \ldots l_p, l'_2 \ldots l'_m} - \lambda^{p-1}_{l_2 \ldots l_p} \lambda^{m-1}_{l'_2 \ldots l'_m}. \tag{2.14}$$

Using equations (2.13) and (2.14), one can easily see that if all vertices are a-vertices,

$$P_a(s, \mathcal{G}) = s^{-L}(s-1)^{N_c} \prod_m [(s-1)^{m-1} - (-1)^{m-1}]^{N^a_m} \tag{2.15}$$

where N^a_m is the number of m-order a-vertices, and where a subscript a has been added to P to emphasise that we are considering only a-vertices. Note that no vertices of order one contribute to equation (2.15). This, of course, is required since $\Sigma(s\delta^{\sigma 1} - 1) = 0$. Equation (2.15) can be cast in a somewhat more useful form if the $m = 0$ term (i.e. the term corresponding to isolated unmarked points) is displayed explicitly. Using the relation $N_t = N_c - N^a_{m=0}$ and the fact that the number of vertices in trees is $\Sigma_{m \geq 2} N^a_m$, we have

$$P_a(s, \mathcal{G}) = (s-1)^{N_t} \prod_{m \geq 3} \left(\frac{(s-1)^{m-1} - (-1)^{m-1}}{s} \right)^{N^a_m}. \tag{2.16}$$

Now consider c-marked vertices. From definition 2, we see that a single c-marked vertex of order $m \geq 2$ can be considered as the sum of two a-vertices of order m and $m - 1$, whereas a singly c-marked vertex of order one is equivalent to an a-vertex of order one. Thus, with each singly marked c-vertex of order $m \geq 2$, there is associated a factor

$$\frac{1}{s}[(s-1)^{m-1} - (-1)^{m-1} + (s-1)^{m-2} - (-1)^{m-2}] = (s-1)^{m-2}. \tag{2.17}$$

In the limit $s \to 1$, the only singly c-marked vertices that survive are of order two and are necessarily external vertices of a tree. This means that in the limit $s \to 1$, g_c marks *only* external vertices whereas g_a can mark internal as well as external vertices, provided all external vertices are marked with g_a.

If we now consider the limit $s \to 1$, $P(s, \mathcal{G})$ becomes

$$P(s, \mathcal{G}) = (s-1)^{N_t}(-1)^{N_v}[1 + O(s-1)] \tag{2.18}$$

where N_v is the number of marked vertices. In deriving equation (2.17) from equation (2.15), we used the relation

$$\sum_{m \geq 3} (m-2)N_m = N_v - 2N_t. \tag{2.19}$$

$P(s, \mathcal{G})$ is unaffected by the presence of any b-vertices. Since $\Sigma_\nu (n\delta^{\nu 1} - 1) = 0$, isolated points cannot be b-vertices. On the other hand, all trees have $\nu = 1$ so that g_b can mark *any* point on any tree with weight one.

Finally, we can write the factor $R(\{\tilde{H}_\nu\}, \{\tilde{h}_\nu\}, \{f_\nu\})$ as

$$R(\{\tilde{H}_\nu\}, \{\tilde{h}_\nu\}, \{f_\nu\}) = (-1)^{N_v} \prod_{i=1}^{N_v} g_\alpha(x_i) \prod_{j=1}^{M} g_b(y_j), \qquad \alpha = \text{a or c} \tag{2.20}$$

where we have adopted the convention that x_i will denote marked vertices y_j b-vertices. Using equations (2.10), (2.19) and (2.20), we can obtain the desired expansion of Z:

$$Z = \sum_{N_v, M} \frac{1}{N_v!} \frac{1}{M!} \sum_{\substack{x_1, \ldots x_{N_v} \\ y_1, \ldots y_M}} \left[g_\alpha(x_1) \ldots g_\alpha(x_{N_v}) g_b(y_1) \ldots g_b(y_M) \right.$$

$$\left. \times K^{N_b} C_{\alpha 0}(N_b, N_t, N_v; x_1, \ldots, x_{N_v}; y_1, \ldots, y_M) \left(\frac{s-1}{n} \right)^{N_t} [1 + o(n, s-1)] \right] \tag{2.21}$$

where $C_{\alpha 0}(N_b, N_t, N_v; x_1, \ldots x_{N_v}; y_1, \ldots, y_M)$ is the number of configurations with N_b bonds, N_t trees, N_v α-marked vertices at $x_1 \ldots x_{N_v}$ and M b-vertices at $y_1 \ldots y_M$. Each configuration is counted only once in this expression. The set of points $x_1 \ldots x_{N_v}$ represents all external vertices for $\alpha = $ c and includes all external points for $\alpha = $ a. The points $y_1 \ldots y_M$ must be vertices through which a tree passes.

3. Applications to polymer problems

In this section, we will apply the results of the preceding section to specific problems in polymer statistics. We will first show how the results of des Cloizeaux (1975) and others for linear polymers can be obtained from this model. We will then consider various statistical problems for branched polymers.

3.1. Linear polymers

For linear polymers, $N_v = 2N_t$. Since $N_v > 2N_t$ for branched trees, the sums in Z can be restricted to linear polymers by choosing $H_\nu^a = [n/(s-1)]^{1/2} H_1 \delta^{\nu 1}$, $h_\nu^b = 0$, $H_\nu^c = 0$ (cf Appendix 1). This gives $g_a = [n/(s-1)]^{1/2} H_1 [1 + O(n^{1/2})]$. In this case, each term in the expansion of Z (equation (2.20)) is proportional to $[n/(s-1)]^{\frac{1}{2}N_v - N_t}$ and only terms with $N_v = 2N_t$ survive the $n \to 0$ limit. Note that since this implies N_v is even, the sign of g_a can be changed without changing any results. This choice for H_ν^a implies that

$$\sum_\nu (\tilde{h}_\nu \tau_b^\nu + f_\nu) = -\tfrac{1}{2} n H_1^2 \delta^{\nu 1} + O(n^{3/2}) \tag{3.1}$$

and that H_{il} (equations (2.5) and (2.6)) satisfies $\Sigma_{il} H_{il}^2 = H_1^2$. With these choices for $\tilde{H}_\nu$, $\tilde{h}_\nu$, n and f_ν, we obtain the expression for Z first obtained by des Cloizeaux (1975):

$$Z_L = \lim_{n \to 0} Z(n, s)$$

$$= \sum_{N_p, N_b} \frac{1}{(2N_p)!} \sum_{x_1, \ldots, x_{2N_p}} H_1(x_1)\ldots H_1(x_{2N_p}) K^{N_b} C(N_p, N_b; x_1, \ldots, x_{2N_p})' \quad (3.2)$$

where $C(N_p, N_b; x_1, \ldots, x_{2N_p})$ is the number of configurations of N_p linear polymers with N_b monomers and end points at $x_1, \ldots, x_{2N_p}$. (We have replaced N_t by N_p since only linear polymers appear.) Since only vertices of order two occur in linear polymers, it is clear from equations (2.15) and (2.16) that equation (3.2) is *completely* independent of the value of s. Two further points are worth noting about equation (3.2). First, only polymers with at least one monomer are included in the sum. If we had chosen $\tilde{h}_\nu = f_\nu = 0$, twice-marked isolated vertices would be included in the sum as is the case in the des Cloizeaux formulation. Second, since all marked vertices are external vertices, the distinction between a-marked and c-marked vertices is irrelevant.

From equation (3.2) follow all of the recently studied properties of the statistics of solutions of linear polymers in the dilute and semi-dilute regimes. In particular, if an additional field $H_2(x)$ along the $\nu = 2$ axis is introduced by writing

$$H_\nu^a = \left(\frac{n}{s-1}\right)^{1/2} (H_1 \delta^{\nu 1} + H_2 \delta^{\nu 2}),$$

then the transverse correlation function

$$G_{22}(x, x') = \left.\frac{\delta^2 Z_L}{\delta H_2(x) \delta H_2(x')}\right|_{H_2=0} = \sum K^{N_b} H_1^{2(N_p-1)} C_{22}(N_b, N_p; x, x') \quad (3.3)$$

where $C_{22}(N_b, N_p; x, x')$ is the number of configurations in which x and x' are the end points of the same polymer. The mean square end-to-end separation $\xi^2(N_b, N_p)$ of a polymer can easily be obtained from C_{22}.

The external fields $g_\nu^b(x)$ are useful whenever it is desired to mark internal vertices. It is often of some experimental interest to know the probability that points x and x' are any two vertices (internal or external) of any polymer. This can be obtained by the addition of the field $g_b(x) = g_1^b(x)$:

$$D_{11}(x, x') = \left.\frac{\delta^2 Z_L}{\delta g_b(x) \delta g_b(x')}\right|_{g_b=0} = \sum K^{N_b} H_1^{2N_p} B_{11}(N_b, N_p; x, x') \quad (3.4)$$

where $B_{11}(N_b, N_p; x, x')$ is the number of configurations in which the points x and x' are any two points on any polymer. Light scattering experiments probe B_{11} directly.

The probability that the points x and x' are any two vertices on the same polymer can be obtained by introducing an external field $g_d(x)$ conjugate to

$$\tau_d(x) = \frac{1}{n-2} [\delta^{\nu(x)1} + (n-1)\delta^{\nu(x)2} - 1] \quad (3.5)$$

It is clear from the definition that

$$\tau_d(x) = \begin{cases} 0 & \text{if } \nu(x) = 1 \\ 1 & \text{if } \nu(x) = 2 \end{cases}$$

and $\Sigma_{\nu(x)} \tau_d(x) = 0$. It is then straightforward to show that

$$D_{22}(x, x') = \frac{2}{H_2^2} \frac{\delta^2 Z_L}{\delta g_d(x) \delta g_d(x')}\Bigg|_{H_2 = g_d = 0} = \sum K^{N_b} H_1^{2(N_p - 1)} B_{22}(N_b, N_p; x, x') \tag{3.6}$$

where $B_{22}(N_b, N_p; x, x')$ is the number of configurations in which the points x and x' are any two vertices on the same polymer. The radius of gyration of a single polymer can be obtained from B_{22}:

$$\langle R_G^2 \rangle = \frac{\Sigma_{x'} |x - x'|^2 B_{22}(N_b, N_p; x, x')}{\Sigma_{x'} B_{22}(N_b, N_p; x, x')}. \tag{3.7}$$

An alternative way of calculating the two functions B_{11} and B_{22} using the n-vector model formalism has been given by Moore (1977) and Schafer and Witten (1977).

As a third example, let us consider the number of configurations $C(N_b, N_p; x_1, x_2; y)$ in which a single polymer with end points at x_1 and x_2 passes through $y \neq x_1 \neq x_2$. This can be obtained via

$$\frac{\delta^3 Z_L}{\delta H_2(x_1) \delta H_2(x_2) \delta g_d(y)}\Bigg|_{H_2 = g_d = 0} = \sum K^{N_b} H_1^{2(N_p - 1)} C(N_b, N_p; x_1, x_2; y). \tag{3.8}$$

Generalisations of this relation to more complicated correlation functions are obvious.

3.2. Branched polymers and solutions of trees

In order to allow for branched configurations of polymers an alternate normalisation of g_a and g_c is needed. Choosing g_a and g_c to be independent of n and s in the $n \to 0$ limit, we obtain

$$Z_B(K, g_\alpha, g_b, \lambda)$$

$$\equiv \lim_{n \to 0} Z(K, g_\alpha, g_b, n, s)\Bigg|_{s-1 = \lambda n}$$

$$= N_v^\Sigma, M\Bigg(\frac{1}{N_v!} \frac{1}{M!} K^{N_b} \lambda^{N_t} \sum_{\substack{x_1, \ldots, x_{N_v} \\ y_1, \ldots, y_M}} g_\alpha(x_1) \ldots g_\alpha(x_{N_v}) g_b(y_1) \ldots g_b(y_M)$$

$$\times C_{\alpha 0}(N_b, N_t, N_v; x_1, \ldots, x_{N_v}; y_1, \ldots, y_M)\Bigg). \tag{3.9}$$

We see, therefore, that $\ln\lambda$ acts as a chemical potential for trees, $\ln K$ for monomers and $\ln g_\alpha$ for end points. If $\alpha = c$, all end points are external vertices. If $\alpha = a$, some end points will be internal vertices. A word of caution regarding equation (3.9) when $\alpha = c$ is perhaps appropriate. From equation (A5), we see that g_c lies between zero and one when $s = 1$ for $H_1^c > 0$. Some problems may occur if it is necessary for g_c to be greater than unity in order, for example, to obtain a specified average number of end points. We stress again that each configuration is counted only once in $C_{\alpha 0}$.

A great deal of information is contained in equation (3.9). In this paper, we are principally interested in the statistics of single branched polymers. We therefore introduce the notation

$$G_\alpha^{k,m}(K, g_\alpha; x_1, \ldots, x_k; y_1, \ldots, y_m)$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \frac{\delta^{k+m} Z_B}{\delta g_\alpha(x_1) \ldots \delta g_b(y_m)}\bigg|_{g_b=0}$$

$$= \sum_{N_b, N_v} K^{N_b} g_\alpha^{N_v-k} C_\alpha^{k,m}(N_b, N_v; x_1, \ldots x_k; y_1, \ldots, y_m) \tag{3.10}$$

where $C_\alpha^{k,m}(N_b, N_v; x_1, \ldots, x_k; y_1, \ldots, y_m)$ is the total number of configurations of a branched tree passing through $y_1, \ldots, y_m$ with N_v end points, k of which are at $x_1, \ldots, x_k$. Two special cases of this function will be of particular interest in the next section. If $g_\alpha = 0$, only those configurations with $N_v = k$ remain. Setting $G_{\alpha 0}^{k,m} = G_\alpha^{k,m}(g_\alpha = 0)$ and $C_{\alpha 0}^{k,m} = C_\alpha^{k,m}(N_v = k)$, we have

$$G_{\alpha 0}^{k,m}(K; x_1, \ldots, x_k; y_1, \ldots, y_m) = \sum_{N_b} K^{N_b} C_{\alpha 0}^{k,m}(N_b; x_1, \ldots, x_k; y_1, \ldots, y_m). \tag{3.11}$$

To obtain the radius of gyration of a branched tree, we use

$$G_\alpha^{0,2}(k, g_\alpha; y, y') = \Sigma K^{N_b} g_\alpha^{N_v} C_\alpha^{0,2}(N_b; N_v; y, y') \tag{3.12}$$

and

$$\langle R_G^2 \rangle = \frac{\Sigma_{y'}(y - y')^2 C_\alpha^{0,2}(N_b, N_v; y, y')}{\Sigma_{y'} C_\alpha^{0,2}(N_b, N_v; y, y')}. \tag{3.13}$$

If N_v is small enough so that there is only one type of tree (cf introduction), as for $N_v \leqslant 5$ with only bi- and tri-functional units, this expression gives the radius of gyration for a polymer as well as a tree. For larger values of N_v, equation (3.13) weights all configurations equally and would give the correct value of $\langle R_G^2 \rangle$ for a polymer only for a special probability distribution in which each type of polymer is weighted by $(N^T)^{-1}$ (cf introduction).

It is possible to discuss polydisperse solutions of trees in which each configuration receives an equal weighting by fixing the average number of monomers, polymers and external vertices via

$$\langle N_b \rangle = K \frac{\delta \ln Z_B}{\delta K}\bigg|_{g_b=0}; \qquad \langle N_t \rangle = \lambda \frac{\delta \ln Z_B}{\delta \lambda}\bigg|_{g_b=0}; \qquad \langle N_v \rangle = g_\alpha \frac{\delta \ln Z_B}{\delta g_\alpha}\bigg|_{g_b=0}$$

$$\tag{3.14}$$

We will consider this and the more realistic ensemble of Zimm and Stockmayer (1949) in a future publication. Finally it is possible to discuss a single branched polymer in a semi-dilute solution of linear polymers or semi-dilute mixtures of linear and branched polymers by choosing

$$H_\nu^\alpha = \delta^{\nu 1}\left(\frac{n}{s-1}\right)^{1/2} H_1 + \delta^{\nu 2} H_2, \qquad \alpha = \text{a, c.}$$

Linear polymers are then marked by H_1 and branched polymers by $g_2^\alpha(H_2)$.

4. Field theory

In this section, we will convert the Hamiltonian, of equation (2.5) into a field theory suitable for use in mean-field calculations and the ϵ-expansion. We will then concentrate on finding scaling forms for the functions $G_\alpha^{k,m}$ and $C_\alpha^{k,m}$ introduced in the previous section. A discussion of other properties will be presented in a later publication.

We begin with a Hubbard–Stratanovitch transformation of equation (2.5). Let $K(x, x')$ be a matrix whose elements are K if x and x' are nearest neighbours and zero otherwise. Then, we have

$$
\exp\!\Big(K \sum_{\langle x,x' \rangle} Q_{il}(x)Q_{il}(x') \Big)
$$

$$
= \int \mathscr{D}\{\Psi\} \Big[\exp\!\Big(-\tfrac{1}{2} \sum_{x,x'} \Psi_{il}(x)K^{-1}(x, x')\Psi_{il}(x') \Big) \exp\!\Big(\sum_{x} \Psi_{il}(x)Q_{il}(x) \Big) \Big] \quad (4.1)
$$

where $\mathscr{D}\{\Psi\} = \Pi_{x,i,l} d\Psi_{il}(x)$.

Using the transformation

$$
\psi_{il}(x) = \sum_{x'} K^{-1}(x, x')\Big(\Psi_{il}(x') + H_{il}(x') \Big) \quad (4.2)
$$

and equations (4.1) and (2.7), we obtain

$$
Z = \exp\!\Big(-\tfrac{1}{2} \sum_{x,x'} H_{il}(x)K^{-1}(x, x')H_{il}(x')
$$

$$
\times \int \mathscr{D}\{\psi\} \exp\!\Big(-\tfrac{1}{2} \sum_{x,x'} \psi_{il}(x)K(x, x')\psi_{il}(x') \Big)
$$

$$
\times \exp\!\Big(\sum_{x} (H_{il}(x)\psi_{il}(x) + \mathscr{S}(x)) \Big) \quad (4.3)
$$

where

$$
e^{\mathscr{S}(x)} = \frac{1}{ns} \sum_{\sigma(x), \nu(x)} \exp\!\Big(\sum_{x'} K(x, x')Q_{il}(x)\psi_{il}(x') \Big) \exp\!\Big(\sum_{\nu} \tilde{h}_\nu(x)\tau_b^\nu(x) \Big) \exp\!\Big(\sum_{\nu} f_\nu(x) \Big).
$$

$$
(4.4)
$$

Equation (4.3) in conjunction with equation (4.4) is an exact expression for the partition function of the Hamiltonian of equation (2.5). If we are concerned only with the critical properties of this model, we can use the long-wavelength continuum form of this equation. First consider the problem when $\tilde{h}_\nu = f_\nu = 0$; then we have

$$
Z = \int \mathscr{D}\{\psi\} \exp[-(\mathscr{H}_1 + \mathscr{H}_B)]. \quad (4.5)
$$

Here $\mathscr{H}_1$ is the Hamiltonian of an isotropic ns-component system:

$$
\mathscr{H}_1 = \int d^d x \ \{\tfrac{1}{2}[r\psi^2(x) + (\nabla\psi(x))^2] + H_{il}(x)\psi_{il}(x) + u_4(\psi^2(x))^2 + u_6(\psi^2(x))^3 + \ldots\}, \quad (4.6)
$$

where $\psi^2 = \Sigma_{il} \psi_{il}^2$, and $\mathcal{H}_B$ is the part of the Hamiltonian that can lead to branching trees:

$$\mathcal{H}_B = -\frac{1}{3!} n w_3 \int d^d x \sum_i \frac{1}{s} \lambda_{l_1 l_2 l_3}^3 \psi_{il_1}(x) \psi_{il_2}(x) \psi_{il_3}(x)$$

$$-\frac{1}{4!} n w_4 \int d^d x \sum_i \frac{1}{s} \lambda_{l_1 l_2 l_3 l_4}^4 \psi_{il_1}(x) \psi_{il_2}(x) \psi_{il_3} \psi_{il_4}(x) + \ldots . \tag{4.7}$$

r is proportional to $(T - T_c)$ where $T_c = T \Sigma_{x'} K(x, x')$, and u_4, u_6, w_3, w_4 etc are proportional to powers of K. The potential w_i obviously give rise to vertices of order i. Note that $\mathcal{H}_B \to 0$ as $n \to 0$, so that the critical properties of this model are controlled by the isotropic $n = 0$ fixed point, as long as there are *a* finite sized polymers with a finite number of branches. This result is, of course expected since we showed in § 3.1 that the present model is identical to the isotropic n-component model in the $n = 0$ limit for the appropriate normalisation of $\tilde{H}$. The formation of an infinite polymer with no closed loops corresponds to tree-like gelation which is described by the zero-state Potts model (Stephen 1976). Thus there is presumably a crossover from the $n = 0$ Heisenberg critical behaviour to that of the zero-state Potts model as the number of monomers and branches is increased. The mathematical mechanism for this crossover is somewhat subtle and will be discussed in a further publication. For the present, we will consider behaviour in the vicinity of the $n = 0$ Heisenberg fixed point.

Under renormalisation group transformations (Wilson and Kogut 1974) where degrees of freedom with wavenumber q satisfying $b^{-1} < |q| < 1$ are removed and the system is rescaled to regain the unit sphere Brillouin zone keeping the coefficient of $(\nabla \psi)^2$ constant, potentials transform according to

$$r' = b^{\lambda_t} r,$$

$$H'_{il}(x) = b^{(d+2-n)/2} H_{il}(bx), \qquad \psi'_{il}(x) = b^{(d-2+n)/2} \psi_{il}(bx)$$

$$w'_i = b^{\lambda_{w_i}} w_i, \qquad u'_i = b^{\lambda_{u_i}} u_i, \tag{4.8}$$

where the exponents $\lambda_t = 1/\nu$, $\lambda_{w_i} \equiv \varphi_{w_i}/\nu$, $\lambda_{u_i} \equiv \varphi_{u_i}/\nu$ and η are evaluated at the isotropic $n = 0$ fixed point. The exponent ν and the correction to scaling (Wegner 1972) exponent λ_{u_4} have been evaluated (Brezin *et al* 1973) to third order in $\epsilon (= 4 - d)$ and η to fourth order in ϵ. They have also been evaluated to a precision of better than 1% using Padé–Borel resummation techniques in three dimensions (Le Guillou and Zinn–Justin 1977). The exponents λ_{w_i} can easily be evaluated to first order in ϵ. We obtain

$$\lambda_{w_p} = 4 - p - \frac{p^2 - 5p + 8}{8} \epsilon. \tag{4.9}$$

We are now in a position to discuss the large-N_b behaviour of the function $C_{a0}^{k,0}(N_b; x_1, \ldots, x_k)$ introduced in § 3.2 (equation (3.11)). First observe that, since

$$\tau_a(x) = \frac{1}{n^{1/2}} e_l^1 \psi_{1l}(x),$$

$$G_{a0}^{k,0} = \lim_{\lambda \to 0} \lim_{n \to 0} \frac{n(-1)^k}{(s-1)n^{k/2}} e_{l_1}^1 \ldots e_{l_k}^1 \langle \psi_{1l_1}(x_1) \ldots \psi_{1l_k}(x_k) \rangle_c \Big|_{(s-1)=\lambda n} \tag{4.10}$$

where $\langle AB \ldots \rangle_c$ signifies the cumulant average of $AB \ldots$. We can see immediately

that the explicit factors of $n^{1/2}$ that accompany the branching potentials w_i (equation (4.7)) are removed by the factor $n^{1-k/2}$ in equation (4.10). Thus, $G_{a0}^{k,0}$ for $k \geqslant 3$ must be explicitly proportional to sums of products of the w_i's. For example, $G_{a0}^{3,0}$ is linearly proportional to w_3 and $G_{a0}^{4,0}$ has a term proportional to w_3^2 and one proportional to w_4 representing branched trees with respectively two vertices of order three and one vertex of order four. For widely separated points, the scaling relations (4.8) imply

$$G_{a0}^{k,0}(r, \{w_i\}; x_1 \ldots x_k) = \xi^{-k(d-2+\eta)/2} F^k\left(\{\xi^{\lambda^{w_i}} w_i\}; \frac{x_1}{\xi}, \ldots, \frac{x_k}{\xi}\right) \tag{4.11}$$

where

$$F^k(\{w_i\}; z_i, \ldots, z_k) = G_{a0}^{k,0}(1, \{w_i\}; z_1, \ldots, z_k) \qquad \text{and} \qquad \xi = (T - T_c)^{-\nu}.$$

Consider now some special cases. $G_{a0}^{3,0}$ must be linearly proportional to w_3 so that

$$G_{a0}^{3,0}(r; x_1, x_2, x_3) = \xi^{-\frac{3}{2}(d-2+\eta)} \xi^{\lambda w_3} f^3\left(\frac{x_1}{\xi}, \frac{x_2}{\xi}, \frac{x_3}{\xi}\right) \tag{4.12}$$

where

$$f^3(z_1, z_2, z_3) = w_3 \frac{\mathrm{d}}{\mathrm{d}w_3} F^3(\{w_i\}; z_1, z_2, z_3)_{\{w_i\}=0}.$$

Then, since

$$C_{a0}^{3,0}(N_b; x_1, x_2, x_3) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{N_b r} G_{a0}^{3,0}(r; x_1, x_2, x_3) \, \mathrm{d}r \tag{4.13}$$

for large N_b, we have

$$C_{a0}^{3,0}(N_b; x_1, x_2, x_3) = N_b^{-1} \xi_p^{-\frac{3}{2}(d-2+\eta)+\lambda_{w_3}} g_a^3\left(\frac{x_1}{\xi_p}, \frac{x_2}{\xi_p}, \frac{x_3}{\xi_p}\right) \tag{4.14}$$

where $\xi_p = N_b^\nu$ and g_a^3 is a function which we will not bother to specify. $G_{a0}^{4,0}$ is a sum of two terms. Following the same procedure as for $G_{a0}^{3,0}$ we obtain

$$C_{a0}^{4,0}(N_b; x_1, x_2, x_3, x_4) = N_b^{-1} \xi_p^{-2(d-2+\eta)+2\lambda_{w_3}} g_{a1}^4\left(\frac{x_1}{\xi_p}, \frac{x_2}{\xi_p}, \frac{x_3}{\xi_p}, \frac{x_4}{\xi_p}\right)$$

$$+ N_b^{-1} \xi_p^{-2(d-2+\eta)+\lambda_{w_4}} g_{a2}^4\left(\frac{x_1}{\xi_p}, \frac{x_2}{\xi_p}, \frac{x_3}{\xi_p}, \frac{x_4}{\xi_p}\right). \tag{4.15}$$

Thus, in the large-N_b limit, the term with two third-order vertices overwhelms the term with one fourth-order vertex since $\lambda_{w_3} \gg \lambda_{w_4}$. Expressions (4.14) and (4.15) are valid for both polymers and trees, though the normalisation of the functions g_a^3, g_{a1}^4 and g_{a2}^4 will be different in two cases.

Now let us return to the field $g_b(x)$ which can mark all vertices in a polymer. Differentiating equation (4.4) with respect to $\tilde{h}_1$, we find the conjugate field

$$\tau_b(x) = \frac{1}{n-1}\left[e^{-\mathcal{S}(x)} \frac{1}{s} \sum_\sigma \exp\left(n^{1/2} \sum_{x'} K(x, x') \psi_{1l}(x') e_l^{\sigma(x)}\right) - 1\right]. \tag{4.16}$$

Expanding in powers of $\psi_{1l}(x)$ and retaining only long-wavelength parts, we obtain

$$e^{\mathscr{S}(x)}\tau_b(x) = \tfrac{1}{2}(zK)^2\left[\sum_l\left(\psi_{1l}^2 - \frac{1}{n-1}\sum_{i\neq 1}\psi_{il}^2\right)\right]$$

$$+\frac{n^{1/2}}{3!}(zK)^3\frac{1}{s}\lambda_{l_1l_2l_3}^3\left(\psi_{1l_1}\psi_{1l_2}\psi_{1l_3} - \frac{1}{n-1}\sum_{i\neq 1}\psi_{il_1}\psi_{il_2}\psi_{il_3}\right)$$

$$+\frac{n}{4!}(zK)^4\frac{1}{s}\lambda_{l_1\ldots l_4}^4\left(\psi_{1l_1}\psi_{1l_2}\psi_{1l_3}\psi_{1l_4} - \frac{1}{n-1}\sum_{i\neq 1}\psi_{il_1}\ldots\psi_{il_4}\right)+\ldots \quad (4.17)$$

where z is the number of nearest neighbours.

Thus it is clear that $\tau_b(x)$ can be written in the form

$$\tau_b(x) = \sum_{m=2}^{\infty} e^{-\mathscr{S}(x)}\tau_{bm}(x) \qquad (4.18)$$

where $\tau_{bm}(x)$ is of order ψ^m and marks a-vertices of order m. Since $e^{-\mathscr{S}(x)} = 1 + O(\psi^2)$, $\tau_b(x)$ can be replaced by its most relevant term

$$\tau_{b2} = \tfrac{1}{2}(zK)^2\sum_l\left(\psi_{1l}^2 - \frac{1}{n-1}\sum_{i\neq 1}\psi_{il}^2\right)$$

in discussions of critical properties at large N_b. τ_{b2} scales like an anisotropy field under renormalisation:

$$\tau_{b2}'(x) = b^{d-\lambda_A}\tau_{b2}(bx) \qquad (4.19)$$

where $\lambda_A = \varphi_A/\nu$ and φ_A is the crossover exponent (Riedel and Wegner 1969, Fisher and Pfeuty 1972) for turning on an anisotropy in the exchange potential K. In the $n = 0$ limit, φ_A is rigorously unity to all orders in perturbation theory (T C Lubensky, unpublished) so that $\lambda_A = 1/\nu$. Thus, when $n = 0$, τ_{b2} and the energy density scale in the same way. Thus, our results for the functions D_{11} and D_{22} (equations (3.4) and (3.6)) agree with those obtained by Moore (1977) and Schafer and Witten (1977). In particular, we predict

$$G_a^{0,2}(r, \{H_{il}\}, \{w_i\}; y, y') = \xi^{2[(1/\nu)-d]}F_a^{0,2}\left(\{\xi^{\lambda_H}H_{il}\}, \{\xi^{\lambda_{w_i}}w_i\}; \frac{y}{\xi}, \frac{y'}{\xi}\right) \qquad (4.20)$$

where $\lambda_H = \tfrac{1}{2}(d + 2 - \eta)$.

Using equation (3.10), we can obtain $C_a^{0,2}$. For example, if we consider only third-order vertices, then

$$C_a^{0,2}(N_b, 3; y, y') = N_b^{-1}\xi_p^{2[(1/\nu)-d]+3\lambda_H+\lambda_{w_3}}g_a^{0,2}\left(3; \frac{y}{\xi_p}, \frac{y'}{\xi_p}\right)$$

$$\qquad (4.21)$$

$$C_a^{0,2}(N_b, 4; y, y') = N_b^{-1}\xi_p^{2[(1/\nu)-d]+4\lambda_H+2\lambda_{w_3}}g_a^{0,2}\left(4; \frac{y}{\xi_p}, \frac{y'}{\xi_p}\right).$$

In both cases $\langle R_G^2\rangle$ is calculated from equation (3.13) is proportional to ξ_p^2 though the constant of proportionality will be different.

Finally, we discuss the correlation function $C_{c0}^{k,0}(N_b; x_1, \ldots, x_k)$ in which all end points are external vertices. From equation (2.3), we have

$$\tau_c(x) = \frac{1}{n^{1/2}}e_i^1\psi_{1i}(x) + \tau_b(x).$$

Thus,

$$G_{c0}^{k,0}(r, \{w_i\}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$$

$$= \lim_{\lambda \to 0} \lim_{n \to 0} \frac{(-1)^k n}{s-1}$$

$$\times \left\langle \left(\frac{1}{n^{1/2}} e_{l_1}^1 \psi_{1 l_1}(\boldsymbol{x}_1) + \tau_b(\boldsymbol{x}_1) \right) \ldots \left(\frac{1}{n^{1/2}} e_{l_k}^1 \psi_{1 l_k}(\boldsymbol{x}_k) + \tau_b(\boldsymbol{x}_k) \right) \right\rangle_c \Big|_{(s-1) = \lambda n}$$

$$= \sum_{\text{permutations}} G_a^{p,k-p}(r, \{w_i\}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_p; \boldsymbol{x}_{p+1}, \ldots, \boldsymbol{x}_k). \qquad (4.22)$$

Restricting our attention to systems with vertices of order three, we see that the dominant term in this series is $G_{a0}^{k,0}$ since $\lambda_{w_3} - \frac{1}{2}(d-2+\eta) > \lambda_A - d$. In other words, if we are only concerned with properties of correlation functions for large N_b, it does not matter whether we specifically mark external vertices or mark both external and internal vertices.

5. Summary

In this paper we introduced an ns-component generalisation of the $2n$-component Hilhorst model. We showed how generating functions for the statistics of branched polymers can be obtained from the thermodymins of this model in the limit $(s-1) = \lambda n$, $n \to 0$. In particular, we showed that the partition function, Z, could be expanded in a power series in the nearest-neighbour interaction strength K, an external field g and λ. Configurations of trees with N_b monomers, N_v external vertices and N_t trees are weighted with a factor $K^{N_b} g^{N_b} \lambda^{N_t}$ in the expansion of Z. This result is summarised in equation (2.21). Thus, we have identified separate chemical potentials for the number of monomers, the number of external vertices and the number of trees.

We showed how the ns-component Hilhorst model can be converted into a field theory. An important result of this operation is the demonstration that all potentials leading to branching are explicitly proportional to some power of n. Thus, even though there is a three-point vertex describing a tri-functional unit, the critical properties of branched polymers are still controlled by the usual $n = 0$ Heisenberg fixed point, as long as there are a finite sized polymers with a finite number of branches. Using this result, we discussed the scaling behaviour of some correlation functions.

The principal purpose of this paper was to discuss the formal properties of the ns-component Hilhorst model. Actual applications of this model to the calculation of the statistics of multiply-branched polymers and the crossover to gelation and vulcanisation will be presented in future publications.

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Appendix 1

In this appendix, we display the forms $\tilde{H}_\nu(x)$, $\tilde{h}_\nu(x)$ and $f_\nu(x)$ must have in order that $e^{-\mathcal{H}}$ can be written as equation (2.8). Let us write

$$
\begin{aligned}
\tilde{H}_\nu(x) &= -H_\nu^a(x) - H_\nu^c(x), \\
\tilde{h}_\nu(x) &= -h_\nu^a(x) + h_\nu^b(x) - h_\nu^c(x) - H_\nu^c(x), \\
f_\nu(x) &= f_\nu^a(x) + f_\nu^b(x) + f_\nu^c(x).
\end{aligned}
\tag{A.1}
$$

We then find that the functions $g_\nu^a(x)$, $g_\nu^b(x)$ and $g_\nu^c(x)$ can be defined via

$$
\begin{aligned}
\exp(-H_\nu^\alpha(x)\tau_\alpha^\nu(x) - h_\nu^\alpha(x)\tau_b^\nu(x) + f_\nu^\alpha(x)) &= (1 - g_\nu^\alpha(x)\tau_\alpha^\nu(x)), \qquad \alpha = a, c \\
\exp(h_\nu^b(x)\tau_b^\nu(x) + f_\nu^b(x)) &= (1 + g_\nu^b(x)\tau_b^\nu(x)),
\end{aligned}
\tag{A.2}
$$

where

$$
g_\nu^a = \frac{\exp(sH_\nu^a) - 1}{1 + (s-1)\exp(sH_\nu^a)}
\tag{A.3}
$$

$$
g_\nu^b = (n-1)\frac{\exp[nh_\nu^b/(n-1)] - 1}{n - 1 + \exp[nh_\nu^b/(n-1)]}
\tag{A.4}
$$

$$
g_\nu^c = \frac{1}{s}[1 - \exp(-sH_\nu^c)],
\tag{A.5}
$$

and

$$
\exp\left(-\frac{n}{n-1}h_\nu^a\right) = \frac{s\exp(-H_\nu^a)}{(s-1) + \exp(-sH_\nu^a)}; \qquad f_\nu^a = -\frac{h_\nu^a}{n-1},
\tag{A.6}
$$

$$
\exp(f_\nu^b) = \frac{n\exp[h_\nu^b/(n-1)]}{(n-1) + \exp[nh_\nu^b/(n-1)]}
\tag{A.7}
$$

$$
\exp\left(-\frac{n}{n-1}h_\nu^c\right) = \frac{s(n-1)\exp[H_\nu^c/(n-1)]}{s(n-1) + 1 - \exp(-sH_\nu^c)}; \qquad f_\nu^c = h_\nu^c.
\tag{A.8}
$$

Note that $g_\nu^\alpha = H_\nu^\alpha + O((H_\nu^\alpha)^2)$, $h_\nu^\alpha = O((H_\nu^\alpha)^2)$ and $f_\nu^\alpha = O((H_\nu^\alpha)^2)$ for $\alpha = a, c$. Also, $g_\nu^b = h_\nu^b + O((h_\nu^b)^2)$ and $f_\nu^b = O((h_\nu^b)^2)$.

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